A solvable model of interface depinning in random media

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Abstract

We study the mean-field version of a model proposed by Leschhorn to describe the depinning transition of interfaces in random media. We show that evolution equations for the distribution of forces felt by the interface sites can be written directly for an infinite system. For a flat distribution of random local forces the value of the depinning threshold can be obtained exactly. In the case of parallel dynamics (all unstable sites move simultaneously), due to the discrete character of the interface heights allowed in the model, the motion of the center of mass is non-uniform in time in the moving phase close to the threshold, and the mean interface velocity vanishes with a square-root singularity.

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1 Introduction

The problem of how a deformable object moves through a heterogeneous medium arises in many different contexts, as exemplified by the title of a recent review of the subject: "Collective transport in random media: from superconductors to earthquakes" [1]. The displacement of a domain wall in a disordered ferromagnet [2], of an interface between two fluids in a porous

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medium [3, 4] or of the contact line of a fluid partially wetting a heterogeneous substrate [5], may be viewed as examples of interfaces with different elastic properties submitted to the competing effects of an external driving force and of local random pinning forces [1, 6, 7, 8].

A general feature of these systems is the existence of a depinning threshold: below a well-defined external force F_c the interface does not move (at zero temperature), due to the collective action of the pinning centers, while for $F > F_c$ it moves with a mean velocity $\bar{v}$. Close to the threshold this velocity has a singular behaviour, analogous to a critical phenomenon:

$$\bar{v} \sim (F - F_c)^{\theta}. \tag{1}$$

Field theory methods such as the dynamic functional renormalization group [6, 10, 11, 12, 13] have been used to predict the dependence of θ and other critical exponents on the space dimensionality, the nature of the disorder and the range of the interactions between parts of the interface. The replica method has also been applied to the problem, indicating that typical pinned interfaces have the same roughness properties as slowly moving ones [14].

Exact results, even for simple models of the depinning transition, have not been obtained so far, and our goal is to show how a class of mean-field models can be solved exactly.

We consider a model first introduced by Leschhorn [15], where the disorder is of the random force type and where space and time are discrete, which appears to give a good qualitative description of that transition [16, 17]. We first show that for the mean-field version of that model dynamical equations can be written exactly in the thermodynamic limit for parallel (i.e., synchronous) dynamics. From these equations of evolution the threshold can be obtained explicitly for some distributions of the random local forces. As an example we study the specific case of a flat distribution and obtain the corresponding value of the threshold as the solution of an algebraic equation. Numerical results are presented for the distance travelled by the interface before stopping, in the pinned phase close to threshold.

We next consider the motion of the interface in the moving phase and find that the mean velocity vanishes as a power law (1), with an exponent $\theta = 1/2$. This value of θ differs from the values found for models with continuous space and continuous relaxational dynamics, for which the mean-field behaviour depends on the form of the pinning potential [9, 10]. We discuss the origin of this difference and relate it to the non-uniformity of the motion very close to the threshold, which is usually not taken into account

in mean-field theories. This non-uniform motion is itself an effect of the discreteness of the allowed heights for the interface.

2 Model and evolution equations

2.1 The Leschhorn models

Different models have been proposed to describe depinning phenomena, with varying degrees of realism, but in view of the very general nature of the problem, it is instructive to study its main qualitative features on the simplest possible systems. In this spirit Leschhorn [15, 16] introduced models discrete in both space and time, in which the interface evolves according to very simple rules. These models are easier and faster to simulate than more realistic continuous models and allow a detailed study of the interface motion near the threshold.

As usual in such studies it is assumed that overhangs may be neglected near the threshold, so a unique height $z_i(t)$ is associated with each site i $(1 \leq i \leq N)$ of the discretized $(d-1)-$dimensional interface. By definition of the model the total force f_i on this site is the sum of the restoring force from the other sites, of an external force F_{ext} and of a random local force:

$$f_i(t) = \sum_j K_{i,j}\Big[z_j(t) - z_i(t)\Big] + F_{ext} + g\ \eta_{i,z_i}\ , \tag{2}$$

and at each time step the interface may move forward at site i only if f_i is positive, otherwise it does not move.

The interactions $K_{i,j}$ in (2) are positive (or zero), so a site which lags behind its neighbours experiences a positive restoring force and it will move unless the pinning force on it is sufficiently negative. The coupling constant g fixes the scale of the random force, while the η_{i,z_i} are uncorrelated random numbers drawn from a given normalized distribution $\rho(\eta)$, a new random number being drawn only if the interface moves at site i (this is what distinguishes pinning models from random growth models described by similar equations, but where a new random number is drawn at each time step).

As will be shown in the following, the time evolution is described by relatively simple equations in the case where all sites experiencing a positive force are updated simultaneously (parallel dynamics) and move by one lattice unit, independently of the magnitude of that force. The equations of motion then read

$$z_i(t+1) = \begin{cases} z_i(t)+1 & \text{if} \quad f_i(t) > 0, \\ z_i(t) & \text{if} \quad f_i(t) \leq 0. \end{cases} \tag{3}$$

As a consequence the instantaneous velocity of the center of mass is just equal to the fraction of sites with a strictly positive force f_i. Another interesting case considered in the literature is that of "extremal dynamics" [18, 19, 20], where at each time step only the site with the largest positive value of f_i moves.

Different interaction kernels $K_{i,j}$ correspond to various physical situations: for an elastic interface $K_{i,j} \neq 0$ only if i and j are neighbouring sites, while in the contact line problem the interaction decays slowly with the distance $(i-j)$ [21, 22]. The model defined in (2) and (3) has been studied numerically by Leschhorn for nearest-neighbour interactions, in the case where η can take two values, $+1$ and -1, with probabilities p and $1-p$ respectively [16, 7]. For $d=2$ (a 1-d interface moving in 2-d space), the results were rather noisy and a large computing effort was needed to obtain an estimate $\theta = 0.25 \pm 0.03$. For $d=3$ the value obtained, $\theta = 0.64 \pm 0.02$, is to be compared with the prediction $\theta = 2/3$ obtained by a first-order expansion about the upper critical dimension, $d_{\rm c} = 5$ for short-range interactions [6].

Remarks:

- A non-zero mean value of the distribution $\rho(\eta)$ has the same effect as an external force $g < \eta >$, so one can assume that $F_{ext} = 0$ without loss of generality. Note that, due to the asymmetry between positive and negative forces in (3), a transition exists even in the absence of an external force and for $\rho(\eta)$ symmetric (i.e., $\rho(\eta) = \rho(-\eta)$). The calculations below are carried out with $F_{ext} = 0$, for notational simplicity, except when indicated.

- The "no-passing" theorem holds [23]: if interface A is everywhere ahead of interface B, i.e., $z_i^A(t) \geq z_i^B(t)$ for all i, then at all their contact points $f_i^A(t) \geq f_i^B(t)$, so B cannot pass A on these sites. At the remaining points B can at best catch up with A, since it moves only one lattice unit at a time. As a consequence moving and static interfaces cannot coexist in the same sample.

- In the thermodynamic limit ($N \to \infty$), the interface moves forward indefinitely for small values of g, and it is pinned for large values of g. In that limit there exists a critical coupling g_c, such that for $g < g_c$ the interface moves with a non-zero mean velocity. The threshold g_c plays the same role

as F_c in (1), and we are interested in what happens for g close to g_c.

2.2 Mean-field evolution equations

The mean-field theory (MFT in the following) is usually identified with the infinite-range limit, where each interface site interacts equally with all the others [31, 3], i.e., the total force on site i is of the form

$$f_i(t) = \bar{z}(t) - z_i(t) + g\ \eta_{i,z_i}, \tag{4}$$

where

$$\bar{z}(t) = \frac{1}{N}\sum_{i=1}^{N} z_i(t) \tag{5}$$

is the average instantaneous position of the interface.

This limit has also been studied numerically [15], for the case where η can take three values $(1, 0, -1)$. It was found that the mean interface velocity vanishes linearly at the depinning threshold, i.e., $\theta = 1$, in agreement with the mean-field prediction for models with discontinuous random forces [6, 9].

We will show now that in this mean-field limit one can write exact evolution equations for $P_k(x,t)$, the fraction of sites at height k and experiencing a local pinning force $g\ x$:

$$P_k(x,t) = \lim_{N\to\infty}\left\{\frac{1}{N}\sum_{i=1}^{N}\delta_{z_i(t),k}\ \delta(x-\eta_{i,z_i})\right\}. \tag{6}$$

It obviously satisfies the normalization condition

$$\sum_k \int_{-\infty}^{\infty} P_k(x,t)dx = 1. \tag{7}$$

Note that we are considering directly quantities defined for the infinite system, thus avoiding the difficulties associated with finite-size effects.

Let us consider the time t when the interface first reaches a given height k. For the newly occupied sites at k, $P_k(x,t)$ is just proportional to $\rho(x)$. At the next time step, for the parallel dynamics studied here, all the sites with $f_i > 0$ move one step forward, so $P_k(x,t)$ gets truncated at $x^* = (k - \bar{z}(t))/g$ (Figure 1). At the same time, among the interface sites located at height $(k-1)$, all those for which $f_i > 0$ move forward, and new random numbers are drawn for those sites, adding to $P_k(x)$ a contribution also proportional to $\rho(x)$, for all x. This construction may be repeated for the following time

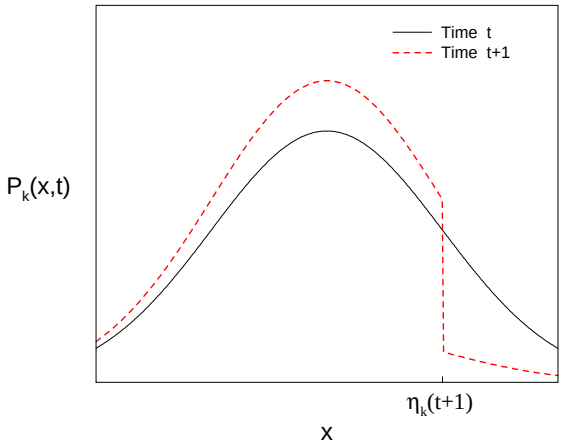

Figure 1: Probability distribution of the forces on the interface sites at height k, at time t (when k is first occupied) and at time $t+1$.

steps, showing that, for all heights k ahead of the region initially occupied by the interface, $P_k(x,t)$ consists of two parts, each of them proportional to $\rho(x)$.

More generally the evolution equations are

$$P_k(x,t+1) = \begin{cases} P_k(x,t) + \rho(x)\int_{\eta_{k-1}(t+1)}^{\infty} P_{k-1}(x',t)dx' & \text{if } x < \eta_k(t+1), \\ \\ \rho(x)\int_{\eta_{k-1}(t+1)}^{\infty} P_{k-1}(x',t)dx' & \text{if } x > \eta_k(t+1), \end{cases} \tag{8}$$

where the discontinuity point varies linearly with k, according to

$$\eta_k(t+1) = \frac{k - \bar{z}(t)}{g} \tag{9}$$

(to avoid the case $x = \eta_k(t+1)$ we assume that $\rho(\eta)$ does not contain delta peaks).

If initially $P_k(x, t=0)$ has the form

$$P_k(x, t=0) = c_k \rho(x), \tag{10}$$

with $\sum_k c_k = 1$ (in particular if initially the interface is flat: $c_k = \delta_{k,0}$), then from (8) $P_k(x,t)$ may be written as

$$P_k(x,t) = \begin{cases} \lambda_k(t)\ \rho(x) & \text{if} \quad x < \eta_k(t), \\ \mu_k(t)\ \rho(x) & \text{if} \quad x > \eta_k(t). \end{cases} \tag{11}$$

Due to this simple structure the evolution equations (8) can be expressed in terms of the $\lambda_k(t)$ and $\mu_k(t)$

$$\mu_k(t+1) = \lambda_{k-1}(t)\int_{\eta_{k-1}(t+1)}^{\eta_{k-1}(t)} \rho(x)dx + \mu_{k-1}(t)\int_{\eta_{k-1}(t)}^{\infty} \rho(x)dx, \quad (12)$$

$$\lambda_k(t+1) = \lambda_k(t) + \mu_k(t+1). \quad (13)$$

The initial conditions are

$$\lambda_k(0) = \mu_k(0) = c_k, \quad (14)$$

and from (13) $\lambda_k(t)$ is a non-decreasing function of time.

Both $\lambda_k(t)$ and $\mu_k(t)$ have a simple physical interpretation. From (12), $\mu_k(t+1)$ is the fraction of sites which jump from height $k-1$ to height k at time $t+1$. Then it is clear from (13) that

$$\lambda_k(t) = \lambda_k(0) + \sum_{t'=1}^{t} \mu_k(t'), \quad (15)$$

so $\lambda_k(t)$ is just the total fraction of sites with a height $\geq k$, if initially all heights larger than k are empty ($c_{k'} = 0$ for $k' > k$).

2.3 A recurrence relation on the λ_k

The evolution equations can be written in a form involving only λ_k, by calculating μ_k from (13) and using that expression in (12). One gets

$$\lambda_k(t+1) - \lambda_{k-1}(t)\int_{\eta_{k-1}(t+1)}^{\infty} \rho(x)dx = \lambda_k(t) - \lambda_{k-1}(t-1)\int_{\eta_{k-1}(t)}^{\infty} \rho(x)dx. \quad (16)$$

The two sides of this equality correspond to the same quantity at two successive times, so it is independent of time and equal to its value at $t = 0$, $\lambda_k(0) (= c_k)$. One obtains

$$\lambda_k(t+1) = c_k + \lambda_{k-1}(t)\int_{\eta_{k-1}(t+1)}^{\infty} \rho(x)dx, \quad (17)$$

where we recall (see (9)) that η_k satisfies

$$\eta_k(t+1) = \frac{k - \bar{z}(t)}{g}. \quad (18)$$

For an interface initially flat and located at $k = k_0$, λ_k is the fraction of sites at heights $\geq k$, so the average interface height is just given by

$$\bar{z}(t) = \sum_{k \geq k_0} k \, [\lambda_k(t) - \lambda_{k+1}(t)], \tag{19}$$

$$= k_0 + \sum_{k > k_0} \lambda_k(t). \tag{20}$$

For the more general initial conditions (14) this relation becomes

$$\bar{z}(t) - \bar{z}(0) = \sum_k \left[\lambda_k(t) - c_k\right]. \tag{21}$$

The instantaneous velocity of the center of mass is thus given by

$$v(t) = \bar{z}(t+1) - \bar{z}(t) = \sum_k \left[\lambda_k(t+1) - \lambda_k(t)\right], \tag{22}$$

a form useful for numerical purposes when this velocity is very small.

Taking the $\lambda_k(t)$ as basic variables, $\bar{z}(t)$ and $\eta_k(t+1)$ can be obtained from (18) and (20) or (21). So, together with (17) and the initial conditions (14), these equations constitute a dynamical system describing the average evolution of an interface, in the thermodynamic limit. This system can be used as such for numerical studies, with the advantage over conventional Monte Carlo simulations that finite-size effects as well as numerical noise are absent. This makes it possible to determine the threshold and to study the critical properties with a much better accuracy.

Remarks:

- For a deformable object like an interface the notion of instantaneous velocity is not unique: For example one may consider the velocity of the center of mass, or of the leading edge, and they are usually different. In the present model the velocity of the leading edge fluctuates strongly, being either 0 or 1, but close to the threshold the time interval between two non-zero values increases. Of course, its time average is identical to the time average of the velocity of the center of mass and it vanishes at the threshold.

- Exact dynamical mean-field equations for an infinite system have been obtained by Eissfeller and Opper [24] for spin glasses, but in their approach the resulting equations still contain a noise term and have to be solved by Monte Carlo simulations.

3 An analytically solvable case

For some distributions $\rho(\eta)$ of the local random forces, the integrals that appear in (17) can be calculated easily, making it possible to push the analytical study further. As a simple example we treat the case of a flat symmetric distribution:

$$\rho(\eta) = \begin{cases} 1/2 & \text{if } -1 \leq \eta \leq 1, \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

(any symmetric flat distribution may be reduced to (23) by rescaling the coupling constant g). For simplicity we consider in the following only a flat interface initially located at $k = 0$.

3.1 Evolution equations

$P_k(x,t)$ is then just made up of two constant parts and is non-zero only for a finite number of values of k, as only a finite number of the $\eta_k(t)$ given by (18) lie in the interval $(-1,1)$ where $\rho(\eta)$ is non-zero.

Consider first the leading occupied edge: it stays at $k = k_{max}(t)$ if the total force (4) on each of its sites is ≤ 0, i.e., if $k_{max}(t) - \bar{z}(t) \geq g$. If not, it moves to $k_{max}(t)+1$, so one has the bounds

$$\bar{z}(t) + g + 1 \; > \; k_{max}(t+1) \; \geq \; \bar{z}(t) + g. \tag{24}$$

As for the trailing edge, its position $k_{min}(t)$ remains fixed as long as its most strongly pinned sites experience a non-positive total force, i.e., if $\bar{z}(t) \leq k_{min}(t) + g$. It moves to $k = k_{min}(t)+1$ if on all its sites $f_i > 0$. These two conditions imply that

$$\bar{z}(t) - g + 1 \; > \; k_{min}(t+1) \; \geq \; \bar{z}(t) - g, \tag{25}$$

and combining (24) and (25) one obtains bounds on the interface width:

$$2g + 1 \geq \; k_{max} - k_{min} \; \geq 2g - 1. \tag{26}$$

The evolution equations (17) become

$$\lambda_{k+1}(t+1) = \begin{cases} \lambda_k(t) \, \frac{1-\eta_k(t+1)}{2} & \text{if} \quad |\eta_k(t+1)| \leq 1, \\ 0 & \text{if} \quad \eta_k(t+1) \geq 1, \\ \lambda_k(t) & \text{if} \quad \eta_k(t+1) \leq -1, \end{cases} \tag{27}$$

with $\lambda_0(t) \equiv 1$ and $\bar{z}(t=0) = 0$ as boundary conditions. The position of the leading edge is such that $\lambda_k(t) = 0$ for $k > k_{max}(t)$, and the position $k_{min}(t)$ of the trailing edge is the largest value of k such that $\lambda_k(t) = 1$ (remember that λ_k is the fraction of sites at heights larger *or equal* to k). The average interface position, eq.(20), may then be expressed as

$$\bar{z}(t) = k_{min}(t) + \sum_{k>k_{min}(t)} \lambda_k(t), \tag{28}$$

a form that will be useful in the following.

3.2 Determination of the depinning threshold

In order to find the threshold g_c we note that very close to it, in the moving phase, the interface velocity is very small (if the transition is continuous, as found numerically), implying from (22) that $\lambda_k(t+1) \simeq \lambda_k(t)$ for all k. One is therefore close to a fixed point of (18), (20) and (27). But if a fixed point exists for a given value of the coupling g, there is a non-moving solution of the evolution equations and according to the no-passing theorem this is incompatible with the assumption that the system is in the moving phase. This shows that the threshold g_c corresponds to the appearance of a fixed point when g is increased: For $g < g_c$, there is no fixed point and the interface moves indefinitely; for $g > g_c$, there is a fixed point and the interface comes to a halt.

3.2.1 Self-consistent equations for the halted interfaces

The halted solutions of the evolution equations satisfy the following self-consistent system:

$$\eta_k = \frac{k - z^*}{g}, \tag{29}$$

$$\lambda_{k+1} = \lambda_k \frac{1-\eta_k}{2} \quad \text{for} \quad |\eta_k| \leq 1, \tag{30}$$

$$z^* = k_{min} + \sum_{k>k_{min}} \lambda_k, \tag{31}$$

with

$$\lambda_k = 1 \quad \text{for} \quad k \leq k_{min}, \qquad \lambda_k = 0 \quad \text{for} \quad k \geq z^* + g + 1. \tag{32}$$

In these equations z^* and k_{min} denote respectively the positions of the center of mass and of the trailing edge of a halted interface, whose distribution of local forces (11) is given by the λ_k and η_k, with $\mu_k = 0$.

Noting that (30) may be written

$$\lambda_k - \lambda_{k+1} = \lambda_k \frac{1+\eta_k}{2}, \tag{33}$$

and using (19) to reexpress z^*, one checks easily that a solution of the above system verifies both the normalization and the self-consistency conditions, which here are simply

$$\sum_k \int_{-\infty}^{\infty} P_k(x,t)dx \quad = \quad \sum_{k \geq k_{min}} \lambda_k \frac{1+\eta_k}{2} \quad = \quad 1, \tag{34}$$

$$\sum_k \int_{-\infty}^{\infty} kP_k(x,t)dx \quad = \quad \sum_{k \geq k_{min}} k\lambda_k \frac{1+\eta_k}{2} \quad = \quad z^*. \tag{35}$$

Other stationary solutions verifying these two conditions but not (30) exist, but they would correspond to other initial conditions and cannot be reached starting from a flat interface.

3.2.2 The threshold

In order to find the halted solutions explicitly, we first note that they are invariant by a translation through an integer number of lattice units, so we can fix $k_{min} = 0$ for simplicity. In addition, the system (29 - 31) can be reduced to an equation for the single variable z^*, as the η_k may be obtained from z^* using (29), then the λ_k from (30) and $\lambda_0 = 1$. Reinjecting these values into (31) yields a self-consistent equation for z^*. The precise form of this equation depends on the width of the halted interface, for which only the bounds (26) are known. A search for solutions can be made for the different possible values of the width, but the effort can be reduced using hints obtained from numerical studies. These indicate that the threshold is very close to $g = 2.38$, and that the width of the halted interface is $(k_{max} - k_{min}) = 5$.

Introducing for convenience the variables

$$y = (1-\eta_0)/2 \quad \text{and} \quad u = 1/2g, \tag{36}$$

equation (29) may be written

$$(1-\eta_k)/2 = y - k\,u, \tag{37}$$

and from (30) and (31) the self-consistency conditions for a halted interface of width 5 read

$$z^* = g\,(2y-1) = \sum_{k=1}^{5} \lambda_k, \tag{38}$$

where the λ_k, considered as functions of y and u, are polynomials of degree k in y, defined by

$$\lambda_1 = y, \qquad \lambda_{k+1} = (y - k\,u)\lambda_k. \tag{39}$$

Eq. (38) finally reduces to a polynomial equation of degree 5 in y :

$$\begin{aligned} R_g(y) &= g + y[1 - 2g - u + 2u^2 - 6u^3 + 24u^4] + y^2[1 - 3u + 11u^2 - 50u^3] \\ &\quad + y^3[1 - 6u + 35u^2] + y^4[1 - 10u] + y^5 \quad = \quad 0. \end{aligned} \tag{40}$$

All the λ_k have to be positive, so acceptable solutions lie in the range $2/g < y < 1$. They exist only if $g \geq g_c$, the value of g for which $R_g(y)$ admits a double root in that range, which we identify with the depinning threshold.

Determining if a polynomial has a double root is a standard problem in algebra [25] and it may be done very accurately. One obtains

$$g_c = 2.38006232..., \tag{41}$$

The other parameters of the critical halted interface are

$$\eta_0 = -0.7990787..., \quad \lambda_1 = y_c = 0.89953936..., \quad z^* = 1.901857... \tag{42}$$

Its density profile is given by the differences $(\lambda_k - \lambda_{k+1})$, which may be deduced from these values using (29) and (30).

Remark: The value of the threshold does not depend on the particular dynamical rules chosen, as long as only unit jumps are allowed and the stopping rule is $f_i \leq 0$. To see this, let us show that a weaker form of the no-passing theorem holds for interfaces with different dynamics. Consider an interface pinned under parallel dynamics: this implies that on all its sites the force $f_i \leq 0$, so it would also be pinned under extremal dynamics. More generally an interface moving under extremal dynamics cannot pass one under parallel dynamics. The converse is not true, but a parallel interface B cannot pass a pinned extremal one A, since at all their contact points $f_i^B \leq f_i^A \leq 0$. This suffices to show that the nature of the phase for a given value of g is the same for both types of dynamics, though some aspects of the critical behaviour might depend on the dynamics considered [26], [27].

3.2.3 Effect of an external force

The case of a non-zero external force can be treated along similar lines. The final result is that the threshold now depends on F_{ext} and is given by the value of g for which the equation

$$R_g(y) + F_{ext} = 0, \tag{43}$$

has a double root. The expression (40) of $R_g(y)$ may change when F_{ext} varies, as it depends on the width of the critical interface, which in turn may depend on F_{ext}. For small F_{ext}, however, we expect the width to remain the same (= 5) and since at the threshold for zero external force

$$R_{g_c(0)}(y_c) = \partial R_{g_c(0)}(y)/\partial y \mid_{y=y_c} = 0, \tag{44}$$

one obtains by expanding (43) a linear dependence of g_c on F_{ext} :

$$g_c(F_{ext}) - g_c(0) \simeq -\frac{F_{ext}}{(\partial R_g/\partial g)^*} \simeq 2.3901...\ F_{ext}\ , \tag{45}$$

where the star symbol denotes a quantity taken at the threshold for $F_{ext} = 0$.

3.3 Stopping distance in the pinned phase

Armed with these exact results, we can now study in detail the behaviour of (18), (27) and (28) in the immediate vicinity of the threshold.

Figure 2 displays numerical results for the distance $z_f(g)$ at which the center of mass of an interface initially located at $k = 0$ stops, in the pinned phase just above g_c. When $g \to g_c$, the data are well fitted by

$$z_f(g) \simeq z^* - c\sqrt{g - g_c}, \tag{46}$$

where z^* is the value (42) obtained for the critical interface (this should be expected since at g_c there exists only one stationary solution with $k_{min} = 0$).

It is also interesting to study the way the interface slows down before stopping. The numerical results show that its velocity vanishes linearly as a function of $\bar{z}(t)$:

$$v(t) \simeq a\ (z_f(g) - \bar{z}(t)). \tag{47}$$

When $v(t)$ is very small it may be replaced by its continuous approximation $d\bar{z}/dt$, and integration of (47) yields an exponential convergence at large times:

$$\bar{z}(t) \simeq z_f(g) - c'\ e^{-at}. \tag{48}$$

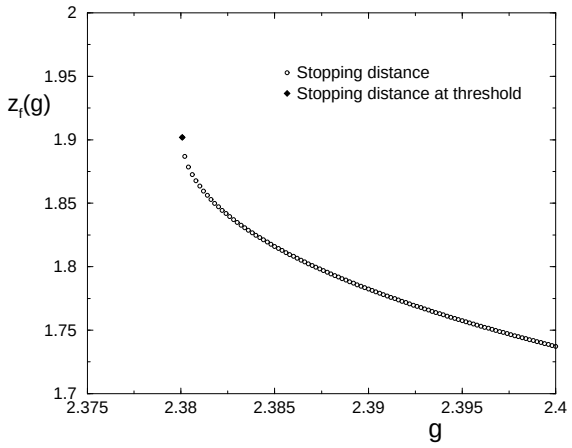

Figure 2: Measured stopping distance $z_f(g)$ for an interface initially flat at position $z = 0$, in the pinned phase, for a uniform distribution of random local forces. The filled diamond indicates the exact values (z^*, g_c) obtained in (41) and (42).

The factor a in (47) depends on g, it vanishes when $g \to g_c$ and right at threshold convergence is found to be much slower. Figure 3 shows $[v(t)]^{1/2}$ versus $\bar{z}(t)$ at $g = g_c$. The data are well fitted by

$$v(t) \simeq b\ (z^* - \bar{z}(t))^2, \tag{49}$$

from which one obtains that asymptotically

$$\bar{z}(t) \simeq z^* - 1/bt. \tag{50}$$

As will be argued in the next section and developed in the Appendix, these results can be simply understood in the framework of a standard saddle-node bifurcation for dynamical systems.

3.4 Moving phase: Numerical results and heuristic arguments

3.4.1 Numerical results

In the moving phase, when the coupling g increases starting from low values, one observes numerically that, for $g < g_c - 0.02$, the mean velocity first decreases linearly with g. But this behaviour changes closer to the threshold

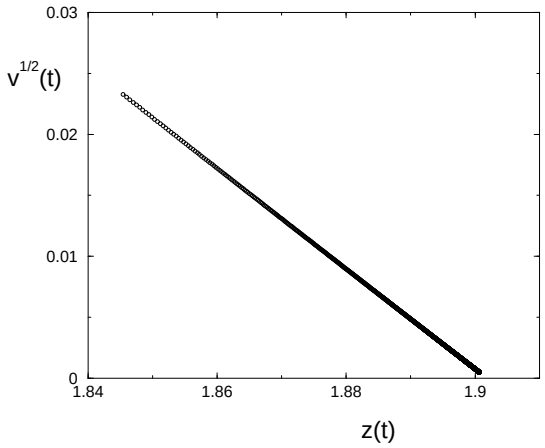

Figure 3: Instantaneous velocity of the center of mass *vs* its position, at the depinning threshold ($g_c = 2.3800623...$).

(Figure 4). Very close to the threshold the time dependence of $v(t)$, the instantaneous velocity of the center of mass, also becomes non-uniform: most of the time the interface creeps very slowly, with sudden bursts during which it moves much more rapidly. The motion looks periodic (Figure 5), with a spatial period of one lattice unit.

The *minimum* velocity of the center of mass vanishes linearly with $(g_c - g)$ up to the threshold, but its *mean* velocity, measured over a time sufficiently long to cover several lattice units, decreases like $(g_c - g)^{1/2}$ (Figure 6). This behaviour would be difficult to observe in a Monte Carlo study of the model defined by (3) and (4), due to finite size effects and to the increasing period. In particular, stopping the simulations after a fixed number of time steps, independently of the distance to g_c, would give incorrect results.

3.4.2 A saddle-node bifurcation

In figure 5 we have seen that most of the time the interface velocity is close to its minimum. Its profile differs then little from the critical one obtained above.

The situation is reminiscent of the one encountered in simple models à la Pomeau-Manneville of intermittency [28]. These consider mappings of the form

$$X(t+1) = F_g(X(t)), \tag{51}$$

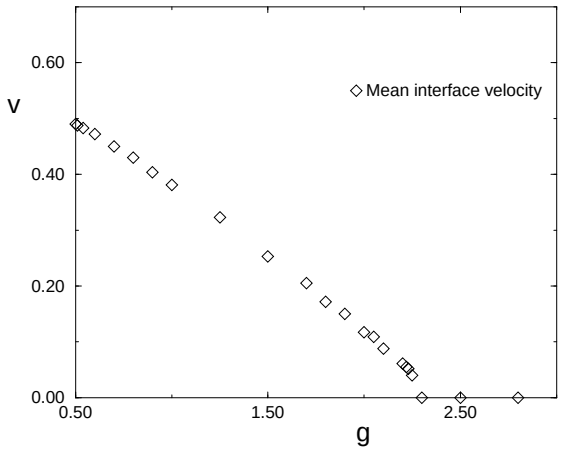

Figure 4: Average interface velocity *vs* g, in the moving phase.

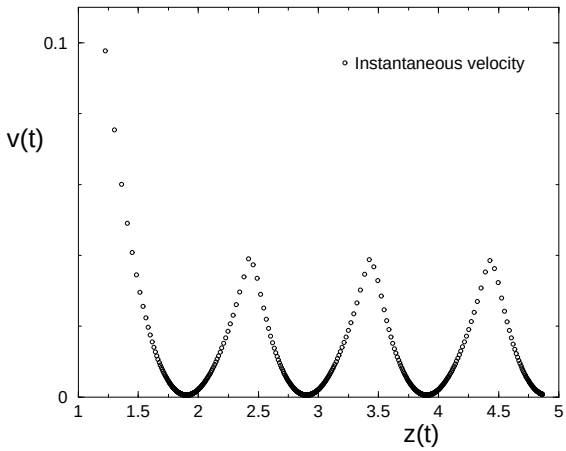

Figure 5: Instantaneous interface velocity *vs* average position, at $g = 2.378$, very close to the threshold.

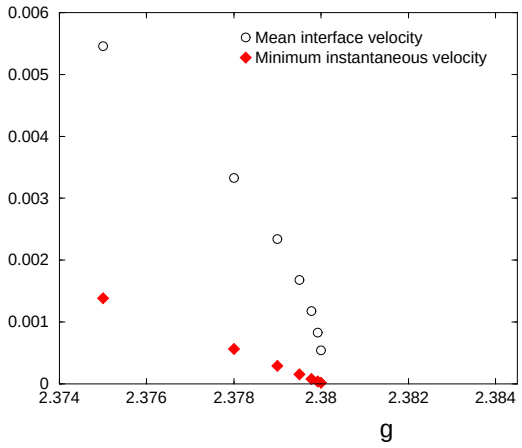

Figure 6: Mean interface velocity and minimum instantaneous velocity, close to the threshold in the moving phase.

and describe the laminar-to-intermittent transition as a saddle-node bifurcation, i.e., the merging of two fixed points of (51) into a double fixed point X^*, at the critical value of the control parameter. In the intermittent phase no fixed point exists, the system spends most of its time in a near-critical laminar regime, where $X(t) \simeq X^*$, with short turbulent bursts during which $X(t)$ varies rapidly.

It is shown in the Appendix that close to threshold the present system may indeed be cast in a form similar to (51): the instantaneous velocity may be expressed to leading orders as a function of the average position $\bar{z}(t)$ only. It can be expanded around a minimum as

$$v(t) = v_{min} + C\ (\bar{z}(t) - z_{min})^2 + \ldots, \tag{52}$$

where the minimum velocity v_{min} is given to leading order by

$$v_{min} = A(g_c - g) + \ldots, \tag{53}$$

z_{min} is a position for which the minimum velocity is reached and the constants A and C can be calculated explicitly (see the Appendix).

The mean velocity $\bar{v}$ may be obtained by integrating (52) over half the period T, assuming that the region of high velocity makes a negligible contribution to the total transit time. One has

$$\frac{T}{2} = \int_{z_{min}}^{z_{min}+1/2} \frac{dz}{v(t)} \simeq \int_{z_{min}}^{z_{min}+1/2} \frac{dz}{v_{min} + C(z - z_{min})^2}\ . \tag{54}$$

Hence, for $v_{min} \ll 1$:

$$\bar{v} = 1/T \simeq (C v_{min})^{1/2}/\pi \tag{55}$$

$$\bar{v} \simeq \frac{\sqrt{AC}}{\pi}(g_c - g)^{1/2}, \tag{56}$$

in agreement with the numerically observed behaviour and yielding the exponent of the mean velocity in (1)

$$\theta = 1/2. \tag{57}$$

4 Discussion and conclusion

We now discuss the result obtained here for the depinning exponent of the infinite-range Leschhorn model and its relation with other sytems. The value $\theta = 1/2$ is the same as for a contact line on a smooth substrate periodically modulated in the direction of motion [29, 30], so the picture which emerges from our results is that very close to threshold the interface dynamics reduces to the motion of the center of mass in a smooth effective washboard-type periodic potential close to the critical tilt angle.

It differs however from the value $\theta = 1$ usually quoted in the literature for the depinning exponent at the upper critical dimension [6, 8], and it is natural to ask what features of the model may explain this difference.

For models with continuous space and continuous relaxational dynamics the mean-field behaviour depends on the analytic properties of the pinning potential [9, 10] - for potentials with random cusps (corresponding to discontinuous random forces such as assumed in the Leschhorn model) the value of the critical exponent is found to be $\theta = 1$. Close to the upper critical dimension the RG analysis indicates that under coarse graining cusp singularities are dynamically generated in the pinning potential, even if they were absent initially [6, 13], so this case of cusped potentials is the natural starting point for an ϵ-expansion.

The origin of the difference with the exact result obtained here may be traced back to a basic assumption usually made in the mean-field theories which is not fulfilled here - namely that in the moving phase the instantaneous velocity $v(t)$ may be replaced in the equations of motion by its mean value over time, which can then be determined as the solution of a self-consistency equation. This assumption was first introduced in the study of the depinning of charge-density waves [31], it amounts to saying that there

is no qualitative difference between a situation where the interface is driven at constant velocity and one where it is submitted to a constant external force. The physical idea is that the fluctuations should average out for a very large system, and it is generally accepted that non-uniformity effects are irrelevant for the critical mean-field behaviour.

On the contrary, in the model studied here, the instantaneous velocity remains strongly non-uniform in the thermodynamic limit (see Figure 5), and the mean and the minimum velocity vanish with different exponents. As shown by the periodicity of the motion, this is related to the discreteness of the allowed positions for the interface, through the constraint of unit moves. In other words a periodic modulation of the potential in which the interface moves may be a relevant perturbation, in the RG sense. This is physically reasonable, as above the upper critical dimension an interface remains flat [6] (in the sense that its mean square width does not diverge with its size), so it cannot effectively average out the underlying periodic potential.

This situation is reminiscent of the pinning of interfaces by the lattice potential in crystalline materials, first studied by Cahn long ago [32]. The assumption of discrete jumps is relevant to various experimental situations where the pinning defects are well separated for instance in dilute magnetic materials or in the motion of contact lines on controlled heterogeneous substrates [33, 34], so the properties of that class of models are interesting in their own. The simultaneous effects of disorder and crystal-lattice pinning on the static roughening of elastic manifolds have been studied by several authors (see [35], [36], [37], [38] and references therein).

For a different model, aimed at describing the motion of visco-elastic interfaces, Marchetti et al. [39] have also observed (numerically) that in the infinite-range limit the velocity fluctuations do not vanish for large system sizes. This came as a surprise, which they attributed to an instability of the constant-v solution in the thermodynamic limit. It is striking that their system shares this type of behaviour with our model, suggesting that the non-uniformity of the velocity may be more general.

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A Appendix: Derivation of (52) and (53)

An expansion of the form (52) for the instantaneous velocity $v(t)$ close to the threshold would follow along well-known lines if the evolution equations could be put under the form of a one-variable relation

$$\bar{z}(t+1) = F_g(\bar{z}(t)), \tag{58}$$

such that the equation $z = F_g(z)$ admits a double root for $g = g_c$. The situation would be similar to the Pomeau-Manneville theory of intermittency [28], in which the transition is described as the merging of two fixed points, a stable and an unstable one, which then disappear when the control parameter g is varied. We show here that for the uniform disorder distribution studied in Section (3), close to the threshold, the equations of motion (17), (18) and (20) can indeed be cast in a form analogous to (58).

Let us consider the time evolution when the instantaneous velocity is very small, so that $k_{min}(t)$ and the interface width may be assumed to remain constant during a large number of consecutive time steps. Eqs (27) and (28) can then be combined in a relation giving the present interface average position from its values at the 5 previous time steps. Using the notations

$$y(t) = y_0 = \frac{1}{2}(1 - \frac{k_{min} - \bar{z}(t)}{g}), \quad y_j = \frac{1}{2}(1 - \frac{k_{min} - \bar{z}(t-j)}{g}), \tag{59}$$

for $j = 1$ to 5, one gets

$$g(2y_0 - 1) = y_1 + y_2(y_1 - u) + y_3(y_2 - u)(y_1 - 2u) + \cdots + y_5(y_4 - u)\ldots(y_1 - 4u), \tag{60}$$

where $u = 1/2g$. This relation is a five-dimensional dynamical system for $y(t)$, it is the time-dependent counterpart of (38) and is linear with respect to each of the y_j. It is exact in the pinned phase near the stopping point. In the moving phase it is valid locally, close to a velocity minimum, and remains valid as long as k_{min} does not change and the width of the interface remains equal to 5.

Since $v(t)$ and its derivatives are assumed to be very small, we may approximate the interface displacements in (60) by

$$y_j(t) - y(t) = -\frac{j}{2g}v(t) + \frac{j(j-1)}{4g}\frac{dv}{dt} + \cdots, \tag{61}$$

where the neglected terms depend on higher-order derivatives of $v(t)$. In the pinned phase ($g > g_c$), (60) admits a fixed point where $y_j \equiv y_c(g)$,

corresponding to the halted interface. Expanding (60) to first order in the small quantities $v(t)$ and $[y_c(g)-y(t)]$, we obtain a linear relation of the form (47), where the proportionality coefficient vanishes for $g = g_c$, as expected from the numerical results in Section 3.

In the moving phase (60) has no fixed point and we have to expand it around the current value of $y(t)$ (note that as we consider the vicinity of a minimum of $v(t)$, this insures that $dv(t)/d(t) << v(t)$). Regrouping the terms independent of v and those linear in v, (60) becomes

$$R_g(y) \simeq \frac{v(t)}{2g} Q_g(y), \tag{62}$$

where $R_g(y)$ is the polynomial appearing in the study of static solutions and is given by eq.(40). $Q_g(y)$ is a polynomial of degree 4 in y, which may be expressed as

$$\begin{aligned} Q_g(y) &= 1 + y(y-u)(\frac{2}{y} + \frac{1}{y-u}) + y(y-u)(y-2u)(\frac{3}{y} + \frac{2}{y-u} + \frac{1}{y-2u}) \\ &\quad + \cdots + y(y-u)\ldots(y-4u)(\frac{5}{y} + \frac{4}{y-u} + \ldots + \frac{1}{y-4u}). \end{aligned} \tag{63}$$

Using the definition of y (59), relation (62) may be cast in the canonical form (58), justifying the claim made in Section (3).

Using the fact that $R_g(y)$ has a double root for $g = g_c$, i.e.,

$$R_{g_c}(y_c) = \partial R_{g_c}(y)/\partial y \mid_{y=y_c} = 0, \tag{64}$$

the leading terms in the expansion of $R_g(y)$ near the threshold, in the range where $y - y_c \simeq (g - g_c)^{1/2}$, read

$$R_g(y) = (\frac{\partial R_g}{\partial g})^*(g - g_c) + \frac{1}{2}(\frac{\partial^2 R_g}{\partial y^2})^*(y - y_c)^2 + \cdots, \tag{65}$$

where for shortness $y_c = y_c(g_c)$ and the star symbol denotes quantities evaluated at the threshold.

Let $\bar{z_c}$ denote the average position of a halted critical interface such that its trailing edge coincides with $k_{min}(t)$. Then $y - y_c = (\bar{z} - \bar{z_c})/2g$, and from (62) and (65) the leading terms in the expansion of $v(t)$ are of the form

$$v(t) = A\ (g_c - g) + C\ (\bar{z}(t) - \bar{z_c})^2 + \ldots \tag{66}$$

The numerical values of the coefficients are

$$A = -(\partial R_g/\partial g)^* \frac{2g_c}{Q^*} = 0.273786... \quad (67)$$

$$C = (\partial^2 R_g/\partial y^2)^* \frac{1}{4g_c Q^*} = 0.167239... \quad (68)$$

where Eq.(66) can be cast in the desired form (52), with the minimum velocity given by

$$v_{min} = A(g_c - g) + \dots \quad (69)$$

as announced in (53). Finally, from (56), the critical behaviour of the mean velocity is

$$\bar{v} \simeq \frac{\sqrt{AC}}{\pi}(g_c - g)^{1/2} \simeq 0.06811\dots\ (g_c - g)^{1/2}, \quad (70)$$

in good agreement with a numerical study very close to threshold.

References

[1] D. S. Fisher, *"Collective transport in random media: from superconductors to earthquakes"*, Physics Reports **301,** 113–150 (1998).

[2] R. Bruinsma, G. Aeppli, Phys. Rev. Letters **52,** 1547 (1984).

[3] J. Koplik, H. Levine, Phys. Rev. B **32,** 280 (1985).

[4] D. Kessler, H. Levine, Y. Tu, Phys. Rev. A **43,** 4551 (1991).

[5] E. Rolley, C. Guthmann, R. Gombrowicz, V. Repain, Phys. Rev. Letters **80,** 2865 (1998).

[6] T. Nattermann, S. Stepanow, L-H. Tang, H. Leschhorn, J. Physique (France) II **2,** 1483 (1992).

[7] H. Leschhorn, T. Nattermann, S. Stepanow, L-H. Tang, Ann. Physik **6,** 1–34 (1997).

[8] M. Kardar, *"Nonequilibrium dynamics of interfaces and lines"*, Physics Reports **301,** 85–112 (1998).

[9] O. Narayan, D.S. Fisher, Phys. Rev B **46,** 11520 (1992).

[10] O. Narayan, D.S. Fisher, Phys. Rev B **48,** 7030 (1993).

[11] L. Balents, D.S. Fisher, Phys. Rev B **48,** 5949 (1993).

[12] D. Ertas, M. Kardar, Phys. Rev. E **49,** 2532 (1994).

[13] P. Chauve, T. Giamarchi, P. Le Doussal Phys. Rev B **62,** 6241 (2000).

[14] A. Hazareesing, M. Mezard, Phys. Rev. E **60,** 1269 (1999).

[15] H. Leschhorn, J. Physics A **25,** L255 (1992).

[16] H. Leschhorn, Physica A **195,** 324 (1993).

[17] M. Jost, Phys. Rev. E **57,** 2634 (1998).

[18] J.-P. Nadal, R.M. Bradley, P. Strenski, J. Physics A **19,** L505 (1986).

[19] S. Roux, A. Hansen, J. Physique I (France) **4,** 515 (1994).

[20] H. Leschhorn, L-H. Tang, Phys. Rev. E **49,** 1238 (1994).

[21] Y. Pomeau, J. Vannimenus, J. Colloid Interface Sci. **104,** 477 (1985).

[22] P.G. de Gennes, Rev. Mod. Phys. **57,** 827 (1985).

[23] A.A. Middleton, D.S. Fisher, Phys. Rev. B **47,** 3530 (1993).

[24] H. Eisfeller, M. Opper, Phys. Rev. Letters **68,** 2094 (1992).

[25] G.A. Korn, T.M. Korn, "Mathematical Handbook for Scientists and Engineers", p. 16–25, McGraw Hill, New York (1961).

[26] M. Paczuski, S. Maslov, P. Bak, Phys. Rev. E **53,** 414 (1996).

[27] O. Narayan, Phys. Rev. E **62,** R7563 (2000).

[28] Y. Pomeau, P. Manneville, J. Physique Lettres **40,** L-609 (1979).

[29] E. Raphael, P.G. de Gennes, J. Chem. Phys. **90,** 7577 (1989).

[30] J.F. Joanny, M.O. Robbins, J. Chem. Phys. **92,** 3206 (1990).

[31] D.S. Fisher, Phys. Rev B **31,** 1396 (1985).

[32] J.W. Cahn, Acta Metall. **8,** 554-562 (1960).

[33] A. Paterson, M. Fermigier, Phys. Fluids **9,** 2210 (1997).

[34] A. Prevost, M. Poujade, E. Rolley, C. Guthmann, Physica B **280,** 80 (2000).

[35] J.P. Bouchaud, A. Georges, Phys. Rev. Letters **68**, 3908 (1992).

[36] T. Emig, T. Nattermann, Eur. Phys. J. B **8,** 525-546 (1999).

[37] A. Hazareesing, J.P. Bouchaud, Phys. Rev. Letters **81**, 5953 (1998).

[38] E.T. Seppälä, M.J. Alava, P.M. Duxbury, Phys. Rev. E **62,** 3230 (2000).

[39] M.C. Marchetti, A.A. Middleton, T. Prellberg, Phys. Rev. Letters **85**, 1104 (2000).